ISING VECTORS IN THE VERTEX OPERATOR ALGEBRA V_Λ^+ ASSOCIATED WITH THE LEECH LATTICE Λ

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Abstract. In this article, we study the Ising vectors in the vertex operator algebra V_Λ^+ associated with the Leech lattice Λ. The main result is a characterization of the Ising vectors in V_Λ^+. We show that for any Ising vector e in V_Λ^+, there is a sublattice $E \cong \sqrt{2}E_8$ of Λ such that $e \in V_E^+$. Some properties about their corresponding τ-involutions in the moonshine vertex operator algebra $V^\natural$ are also discussed. We show that there is no Ising vector of σ-type in $V^\natural$. Moreover, we compute the centralizer $C_{\mathrm{Aut}\,V^\natural}(z, \tau_e)$ for any Ising vector $e \in V_\Lambda^+$, where z is a $2B$ element in $\mathrm{Aut}\,V^\natural$ which fixes V_Λ^+. Based on this result, we also obtain an explanation for the $1A$ case of an observation by Glauberman-Norton (2001), which describes some mysterious relations between the centralizer of z and some $2A$ elements commuting z in the Monster and the Weyl groups of certain sublattices of the root lattice of type E_8 .

1. Introduction

The study of vertex operator algebras (VOAs) as modules of the rational Virasoro VOA $L\left(\frac{1}{2},0\right)$ was first initiated by Dong, Mason and Zhu [6]. Partially motivated by their work, Miyamoto [13] introduced the notion of an Ising vector (i.e., a Virasoro vector of weight 2 which generates a copy of the rational Virasoro VOA $L\left(\frac{1}{2},0\right)$ inside a VOA). He also developed a simple method to construct involutions of the automorphism group from Ising vectors. Such an automorphism is often called a Miyamoto involution or a τ-involution and is very useful for studying the automorphism groups of VOAs. Therefore, it is really natural to characterize all Ising vectors in VOAs. For example, all Ising vectors in VOAs associated with binary codes and the VOAs $V_{\sqrt{2}R}^+$ associated with root lattices R were described in [11]. It was also conjectured that if L is an even lattice without roots, then

C. H. Lam is partially supported by NSC grant 95-2115-M-006-013-MY2 of Taiwan.

H. Shimakura is partially supported by the Japan Society for the Promotion of Science Research Fellowships for Young Scientists.

any Ising vector in V_L^+ is contained in a subVOA V_U^+ for a sublattice U of L isomorphic to $\sqrt{2}A_1$ or $\sqrt{2}E_8$.

The moonshine VOA $V^\natural$ constructed by Frenkel-Lepowsky-Meurman [7] is one of the most important examples of VOAs, which can be written as

$$V^\natural = V_\Lambda^+ \oplus V_\Lambda^{T,+},$$

where V_Λ^+ is the fixed point subVOA of the Leech lattice VOA V_Λ by an automorphism θ induced from the isometry $\beta \mapsto -\beta, \beta \in \Lambda$, V_Λ^T is the unique irreducible θ-twisted module of V_Λ and $V_\Lambda^{T,+}$ is the fixed point subspace of V_Λ^T by an automorphism induced from θ. By the construction, there is a natural involution $z \in \mathrm{Aut}\, V^\natural$ such that $z|_{V_\Lambda^+} = 1$ and $z|_{V_\Lambda^{T,+}} = -1$ and this automorphism z belongs to the $2B$ conjugacy class of the Monster group.

On the moonshine VOA $V^\natural$, Miyamoto showed that τ-involutions correspond to the $2A$ involutions of the Monster group and that there is a one to one correspondence between $2A$-involutions of the Monster group and Ising vectors in $V^\natural$ by using the results in [1]. This gives an approach to explain some mysterious phenomena associated with $2A$-involutions of the Monster group by using the theory of VOAs. For example, the McKay observation on the affine E_8-diagram has been studied in [12] using Miyamoto involutions. In this perspective, the explicit description of Ising vectors in $V^\natural$ may provide some new insights on these phenomena.

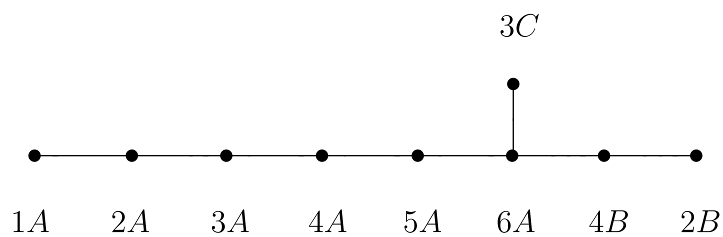
In this article, we shall study the Ising vectors in the VOA V_Λ^+. The main result is a characterization of all Ising vectors in V_Λ^+. In particular, the conjecture in [11] on a characterization of Ising vectors in V_L^+ holds when L is isomorphic to the Leech lattice Λ. In addition, some properties about the corresponding τ-involutions of these Ising vectors in the moonshine VOA $V^\natural$ will be discussed. We also show that there is no Ising vector of σ-type in $V^\natural$. In other words, the involution τ_e is non-trivial for any Ising vector $e \in V^\natural$. Moreover, we compute the centralizer $C_{\mathrm{Aut}\, V^\natural}(z, \tau_e)$ of z and the τ-involution τ_e in the automorphism group of $V^\natural$ for an Ising vector $e \in V_\Lambda^+$. It comes out that if e is obtained from some sublattice $E \cong \sqrt{2}E_8$ in the Leech lattice, then the centralizer $C_{\mathrm{Aut}\, V^\natural}(z, \tau_e)$

also stabilizes the subVOA V_E^+ and $C_{\mathrm{Aut}\,V^\natural}(\tau_e, z)$ acts on V_E^+ as the orthogonal simple group $\Omega^+(8,2)$.

It is well-known that the products of any two $2A$-involutions of the Monster simple group $\mathbb{M}$ fall into one of the following nine conjugacy classes [1, 2]:

$$1A,\ 2A,\ 3A,\ 4A,\ 5A,\ 6A,\ 4B,\ 2B,\ \text{or } 3C.$$

John McKay observed there is an interesting correspondence between these nine conjugacy classes of $\mathbb{M}$ and the nine nodes of the extended Dynkin diagram $\hat{E}_8$ as follows:



In Glauberman-Norton [9], several other patterns and mysterious properties related to the Dynkin diagram were discussed. Among other things, they noted that the centralizer of a certain subgroup generated by two $2A$-involutions and one $2B$-involution in the Monster simple group appears to have a quotient which is isomorphic to the "half" of the Weyl group of the sub-diagram of $\hat{E}_8$ with the relevant node removed.

In our case, the $2A$-involutions are τ_{e_E} and τ_{e_E} and the $2B$-involution is z. Then, $\tau_e\tau_e = 1$ is of the class $1A$. The subdiagram with the $1A$ node removed is of the type E_8 and the Weyl group is of the shape $2.\Omega^+(8,2).2$. By our method, the centralizer $C_{\mathrm{Aut}\,V^\natural}(z, \tau_e)$ stabilizes the subVOA V_E^+ and acts on V_E^+ as $\Omega^+(8,2)$, which is the quotient of the commutator subgroup of the Weyl group of E_8 by its center. It explains the $1A$ case of the observation by Glauberman and Norton. We believe that this is a first step toward a complete understanding of the observation of Glauberman and Norton.

Acknowledgements. Part of the work was done when the second author was visiting the National Center for Theoretical Sciences, Taiwan on January 2007. He thanks the staff of the center for their help.

Notation. Let Ω denote the set $\{1, 2, \ldots, 24\}$. We view the power set $\mathcal{P}(\Omega)$ of Ω as a 24-dimensional vector space over $\mathbb{F}_2$ naturally. For a lattice L, let $L(m)$ denote the set of vectors in L of squared norm $2m$, namely $L(m) = \{v \in L \mid \langle v, v\rangle = 2m\}$.

2. Virasoro VOA and Ising vectors

In this section, we shall recall some basic facts about Virasoro VOAs and Ising vectors. We shall also review the construction of Miyamoto automorphisms.

Let $\mathrm{Vir} = \left(\bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n\right) \oplus \mathbb{C}\mathbf{c}$ be the Virasoro algebra. Then L_n's satisfy the famous commutator relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}\mathbf{c}.$$

Let $L(c, h)$ be the unique irreducible highest weight module of Vir with central charge c and highest weight h. That means $L(c, h)$ is generated by a vector v such that $L_n v = 0$ for $n > 0$, $L_0 v = hv$ and $\mathbf{c}v = cv$. It was shown by Frenkel-Zhu [8] that $L(c, 0)$ is a simple vertex operator algebra. If $c = c_m = 1 - \frac{6}{(m+2)(m+3)}$, $m = 1, 2, 3, \ldots$, then $L(c_m, 0)$ has a unitary form and it is a rational VOA, that means $L(c_m, 0)$ has only finitely many irreducible modules and all $L(c_m, 0)$-modules are completely reducible (cf. [6, 16]).

When $c = c_1 = \frac{1}{2}$, the VOA $L(\frac{1}{2}, 0)$ has exactly three inequivalent irreducible modules, namely, $L\left(\frac{1}{2}, 0\right), L\left(\frac{1}{2}, \frac{1}{2}\right)$ and $L\left(\frac{1}{2}, \frac{1}{16}\right)$, and its fusion rules are given as

$$\begin{aligned}
L\left(\frac{1}{2}, \frac{1}{2}\right) \times L\left(\frac{1}{2}, \frac{1}{2}\right) &= L\left(\frac{1}{2}, 0\right), \\
L\left(\frac{1}{2}, \frac{1}{2}\right) \times L\left(\frac{1}{2}, \frac{1}{16}\right) &= L\left(\frac{1}{2}, \frac{1}{16}\right), \\
L\left(\frac{1}{2}, \frac{1}{16}\right) \times L\left(\frac{1}{2}, \frac{1}{16}\right) &= L\left(\frac{1}{2}, 0\right) + L\left(\frac{1}{2}, \frac{1}{2}\right)
\end{aligned}$$

and $L\left(\frac{1}{2}, 0\right)$ acts as an identity.

Definition 2.1. A weight 2 element $e \in V_2$ is called an *Ising vector* if the vertex subalgebra VA(e) generated by e is isomorphic to the simple Virasoro VOA $L(\frac{1}{2}, 0)$ and e is the Virasoro element of VA(e).

Next we shall review the definition of Miyamoto involutions. Let e be an Ising vector. Since VA(e) $\cong$ L$(\frac{1}{2},0)$ is rational, we have the isotypical decomposition

$$V = V_e(0) \oplus V_e(\frac{1}{2}) \oplus V_e(\frac{1}{16}),$$

where $V_e(h)$ denotes the sum of all irreducible VA(e)-submodules of V isomorphic to $L(\frac{1}{2},h)$, $h = 0, \frac{1}{2}, \frac{1}{16}$. An Ising vector e is said to be *of σ-type* if $V_e(\frac{1}{16}) = 0$.

The following result was first proved by Miyamoto [13]:

Theorem 2.2. *Let V be a VOA and $e \in V$ an Ising vector. Then the linear map $\tau_e : V \to V$ defined by*

$$\tau_e = \begin{cases} 1 & \text{on } V_e(0) \oplus V_e(\frac{1}{2}) \\ -1 & \text{on } V_e(\frac{1}{16}) \end{cases}$$

is an automorphism of V.

If $\tau_e = \mathrm{id}$, namely e is of σ-type, then the linear map $\sigma_e : V \to V$ defined by

$$\sigma_e = \begin{cases} 1 & \text{on } V_e(0) \\ -1 & \text{on } V_e(\frac{1}{2}) \end{cases}$$

is an automorphism of V.

Remark 2.3. Note that the automorphism τ_e can also be defined by

$$\tau_e = \exp(16\pi i e_1),$$

where $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $Y(e,z) = \sum_{n\in\mathbb{Z}} e_n z^{-n-1}$, $e_n \in \mathrm{End}\, V$.

3. Lattice VOA and its $\mathbb{Z}_2$-orbifold

Our notation for lattice VOAs here is standard (cf. [7]). Let L be a positive definite even lattice with inner product $\langle\, \cdot\, ,\, \cdot\, \rangle$. Then the VOA V_L associated with L is defined to be $M(1) \otimes \mathbb{C}\{L\}$. More precisely, let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ be an abelian Lie algebra and $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ its affine Lie algebra. Let $\hat{\mathfrak{h}}^- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]$ and let $S(\hat{\mathfrak{h}}^-)$ be the symmetric algebra of $\hat{\mathfrak{h}}^-$. Then $M(1) = S(\hat{\mathfrak{h}}^-) = \mathbb{C}[\alpha(n) \mid \alpha \in \mathfrak{h}, n < 0] \cdot 1$ is the unique irreducible $\hat{\mathfrak{h}}$-module such that $\alpha(n) \cdot 1 = 0$ for $\alpha \in \mathfrak{h}$, $n \geq 0$ and $K = 1$, where

$\alpha(n) = \alpha \otimes t^n$. Moreover, $\mathbb{C}\{L\} = \mathrm{span}_{\mathbb{C}}\{e^\alpha \mid \alpha \in L\}$ is a twisted group algebra of the additive group L such that $e^\alpha e^\beta = (-1)^{\langle \alpha,\beta\rangle} e^\beta e^\alpha$. Note that $M(1)$ is a subVOA of V_L.

The twisted group algebra $\mathbb{C}\{L\}$ can be described by using central extension. Let $\langle\kappa\rangle$ be a cyclic group of order 2 and

$$1 \longrightarrow \langle\kappa\rangle \longrightarrow \hat{L} \xrightarrow{-} L \longrightarrow 1$$

a central extension of L by $\langle\kappa\rangle$ with the commutator map $c_0(\alpha,\beta) = \langle\alpha,\beta\rangle \mod 2$ for any $\alpha,\beta \in L$. Let $L \to \hat{L}, \alpha \mapsto e^\alpha$ be a section. Then the twisted group algebra

$$\mathbb{C}\{L\} \cong \mathbb{C}[\hat{L}]/(\kappa+1) = \mathrm{span}_{\mathbb{C}}\{e^\alpha \mid \alpha \in L\},$$

where $\mathbb{C}[\hat{L}]$ is the usual group algebra of the group $\hat{L}$.

Let $\theta : \hat{L} \to \hat{L}$ be an automorphism of $\hat{L}$ defined by $\theta(a) = a^{-1}\kappa^{\langle\bar{a},\bar{a}\rangle/2}$. Then it induces an automorphism of V_L by

$$\theta(\alpha_1(-n_1)\cdots\alpha_k(-n_k)e^\alpha) = (-1)^k \alpha_1(-n_1)\cdots\alpha_k(-n_k)\theta(e^\alpha).$$

Let $V_L^+ = \{v \in V_L \mid \theta(v) = v\}$ be the fixed subspace of θ in V_L. Then V_L^+ is a subVOA and is often called a $\mathbb{Z}_2$-*orbifold*.

For any sublattice E of L, let $\mathbb{C}\{E\} = \mathrm{span}_{\mathbb{C}}\{e^\alpha \in \mathbb{C}\{L\} \mid \alpha \in E\}$ be a subalgebra of $\mathbb{C}\{L\}$ and let $\mathfrak{h}_E = \mathbb{C}\otimes_{\mathbb{Z}} E$ be a subspace of $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{Z}} L$. Then the subspace $S(\hat{\mathfrak{h}}_E^-)\otimes\mathbb{C}\{E\}$ forms a subVOA of V_L and it is isomorphic to the lattice VOA V_E.

Next, we shall review the constructions of some Ising vectors in a lattice VOA V_L. The action of the corresponding τ-involutions will also be discussed. Recall that two elements e, f in a VOA are said to be *mutually orthogonal* if $Y(e,z)f = Y(f,z)e = 0$.

Theorem 3.1 (cf. [6]). *Let $\alpha \in L(2)$. Then the elements $\omega^+(\alpha)$ and $\omega^-(\alpha)$ defined by*

$$\omega^\pm(\alpha) = \frac{1}{16}\alpha(-1)^2\cdot\mathbf{1} \pm \frac{1}{4}(e^\alpha + \theta(e^\alpha))$$

are two mutually orthogonal Ising vectors.

The following proposition is easy:

Proposition 3.2. *Let $\alpha \in L(2)$. Then the automorphisms $\tau_{\omega^+(\alpha)}$ and $\tau_{\omega^-(\alpha)}$ of V_L are equal to φ_α given by*

$$\varphi_\alpha(u \otimes e^\beta) = (-1)^{\langle \alpha, \beta\rangle} u \otimes e^\beta \qquad \text{for } u \in M(1) \text{ and } \beta \in L.$$

If L contains a sublattice isomorphic to $\sqrt{2}E_8$, then there is another class of Ising vectors in V_L.

Theorem 3.3 (cf. [5])**.** *Let $E \subset L$ be a sublattice isomorphic to $\sqrt{2}E_8$. Then the Virasoro element of V_E is given by*

$$\omega' = \frac{1}{240} \sum_{\alpha \in E(2)} \alpha(-1)^2 \cdot \mathbf{1}$$

and the element e_E defined by

$$e_E = \frac{1}{16}\omega' + \frac{1}{32} \sum_{\alpha \in E(2)} (e^\alpha + \theta(e^\alpha))$$

is an Ising vector.

Remark 3.4. Note that the Ising vectors $\omega^\pm(\alpha)$ and e_E are clearly contained in the orbifold VOA V_L^+

The following theorem can be found in [12].

Theorem 3.5. *Let $E \cong \sqrt{2}E_8$. Then τ_{e_E} is equal to θ in* $\operatorname{Aut} V_E$*. In particular, $\tau_{e_E} = \mathrm{id}$ on V_E^+.*

Now let L be a positive definite even lattice and let

$$L^* = \{\alpha \in \mathbb{Q} \otimes_{\mathbb{Z}} L \mid \langle \alpha, \beta \rangle \in \mathbb{Z} \text{ for all } \beta \in L\}$$

be the dual lattice of L. For $x \in L^*$, define a $\mathbb{Z}$-linear map $\varphi_x : L \to \mathbb{Z}_2$ by

$$\varphi_x(y) = \langle x, y \rangle \qquad \mod 2.$$

Clearly the map

$$\begin{aligned} \varphi : L^* &\longrightarrow \operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Z}_2) \\ x &\longmapsto \ \varphi_x \end{aligned}$$

is a surjective group homomorphism and $\ker\varphi = 2L^*$. Hence, we have $\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z}_2) \cong L^*/2L^*$. For $\alpha \in L^*$, φ_α induces an automorphism of V_L given by

$$\varphi_\alpha(u\otimes e^\beta) = (-1)^{\langle\alpha,\beta\rangle} u\otimes e^\beta \quad \text{for } u \in M(1) \text{ and } \beta \in L.$$

Now suppose $L = E \cong \sqrt{2}E_8$. Then $E^* = \frac{1}{2}E$ and $E^*/2E^* = \frac{1}{2}E/E \cong E_8/2E_8$. For $x \in E^* = \frac{1}{2}E$,

$$\varphi_x(e_E) = \frac{1}{16}\omega' + \frac{1}{32}\sum_{\alpha\in E(2)}(-1)^{\langle x,\alpha\rangle}(e^\alpha + \theta(e^\alpha))$$

is also an Ising vector in V_E^+ and we have $256(=2^8)$ Ising vectors of this form. Since φ_x commutes θ, $\tau_{\varphi_x(e_E)} = \theta$ on V_E and $\tau_{\varphi_x(e_E)} = \mathrm{id}$ on V_E^+ also.

Remark 3.6. It was shown in [3] that $E_8/2E_8$ contains one class represented by 0, 120 classes represented by a pair of roots $\pm\alpha$, and 135 classes represented by 16 vectors forming a 4-frame of E_8, i.e., a subset $\{\pm\alpha_1,\dots,\pm\alpha_8\} \subset E_8$ such that $\langle\alpha_i,\alpha_j\rangle = 4\delta_{i,j}$, $i,j = 1,\dots,8$.

4. Leech Lattice and automorphism group of V_Λ^+

In this section, we will review the automorphism group of Λ and V_Λ^+ for the Leech lattice Λ.

4.1. **Leech lattice and automorphisms.** In this subsection, we will review some basic properties of the Leech lattice Λ and its automorphism group Co_0. Our notation mainly follows that of Conway-Sloane [3].

Let $\mathcal{C} \subset P(\Omega)$ be the extended binary Golay code of length 24. A codeword $\mathcal{O} \in \mathcal{C}$ is called an *octad* if $|\mathcal{O}| = 8$ and a *dodecad* if $|\mathcal{O}| = 12$. The automorphism group of the Golay code $\mathcal{C}$ is the Mathieu group M_{24} and it acts transitively on octads. Note also that $\mathcal{C}$ is generated by octads and there are exactly 759 octads in $\mathcal{C}$.

Let $\{e_i \mid i \in \Omega\}$ be an orthogonal basis of $\mathbb{R}^{24}$ of squared norm 2.

Theorem 4.1 ([3]). *The Leech lattice Λ is a lattice of rank 24 generated by the vectors:*

$$\begin{aligned} &\frac{1}{2}e_X\ ; \quad X \text{ is a generator of the Golay code } \mathcal{C},\\ &\frac{1}{4}e_\Omega - e_1\ ,\\ &e_i \pm e_j\ ,\ i,j \in \Omega \end{aligned}$$

where $e_X = \sum_{i\in X} e_i$.

Definition 4.2. A set of vectors $\{\pm\alpha_1, \dots, \pm\alpha_{24}\} \subset \Lambda$ is called an *n-frame* of Λ if $\langle \alpha_i, \alpha_j\rangle = n\delta_{i,j}$ for all $i,j \in \{1,\dots,24\}$.

For example, $\{\pm 2e_1, \dots, \pm 2e_{24}\}$ is an 8-frame of Λ and $\{\pm(e_{2i-1} \pm e_{2i}) \mid i = 1,\dots,12\}$ is a 4-frame of Λ.

Lemma 4.3. (cf. [3]) *Any vector in $\Lambda(2)$ is contained in a 4-frame of Λ.*

Proof. Since $\{\pm(e_{2i-1} \pm e_{2i}) \mid i = 1,\dots,12\}$ is a 4-frame of Λ and the automorphism group Co_0 of Λ acts transitively on $\Lambda(2)$, we have the desired result. □

Note that the Leech lattice Λ has no vectors of squared norm 2 and is generated by the vectors of squared norm 4.

Let $\mathcal{F} = \{\pm 2\alpha_1, \dots, \pm 2\alpha_{24}\}$ be an 8-frame of Λ. Then $\{\alpha_1, \alpha_2, \dots, \alpha_{24}\}$ is an orthogonal basis of $\mathbb{R}^{24}$ such that $\langle \alpha_i, \alpha_j\rangle = 2\delta_{i,j}$.

Denote

$$\mathcal{C}_\mathcal{F} = \left\{ S \subset \Omega = \{1,2,\dots,24\} \;\middle|\; \frac{1}{2}\sum_{i\in S} \alpha_i \in \Lambda \right\}.$$

It is well-known (cf. [3, 10]) that $\mathcal{C}_\mathcal{F}$ is isomorphic to the binary Golay code $\mathcal{C}$. The Leech lattice Λ will then be generated by the vectors of the form

$$2\alpha_i, \qquad \alpha_i \pm \alpha_j, \qquad \frac{1}{2}\alpha_S = \frac{1}{2}\sum_{i\in S}\alpha_i, \qquad \text{and} \qquad \frac{1}{4}\alpha_\Omega - \alpha_i,$$

where $i,j \in \Omega$ and $S \in \mathcal{C}$.

For any permutation π of Ω, π defines an isometry of $\mathbb{R}^{24}$ by $\pi(\alpha_i) = \alpha_{\pi i}$. If $\pi(\mathcal{C}_\mathcal{F}) = \mathcal{C}_\mathcal{F}$, then π also defines an automorphism of Λ. Let S be a subset of Ω. Then we can also define an isometry $\varepsilon_S^\mathcal{F} : \mathbb{R}^{24} \to \mathbb{R}^{24}$ by $\varepsilon_S^\mathcal{F}(\alpha_i) = -\alpha_i$ if $i \in S$ and $\varepsilon_S^\mathcal{F}(\alpha_i) = \alpha_i$ if $i \notin S$.

The involutions in the Conway group Co_0 can be characterized as follows:

Theorem 4.4 ([3, 10])**.** *There are exactly 4 conjugacy classes of involutions in Co_0. They correspond to the involutions $\varepsilon_S^{\mathcal{F}}$, where $\mathcal{F}$ is an 8-frame and $S \in \mathcal{C}$ is an octad, the complement of an octad, a dodecad, or the set Ω. Moreover, the sublattice $\{v \in \Lambda \mid \varepsilon_S^{\mathcal{F}}(v) = -v\}$ is isomorphic to $\sqrt{2}E_8$, BW_{16}, $\sqrt{2}D_{12}$ and Λ respectively, where BW_{16} is the Barnes-Wall lattice of rank 16.*

Note that the center of the Conway group Co_0 is the cyclic group $\langle \pm 1\rangle$. The quotient group $Co_1 = Co_0/\langle \pm 1\rangle$ is a simple group and it has 3 conjugacy classes of involutions, namely $2A$, $2B$, and $2C$ (cf. [2]).

Proposition 4.5 ([2, 3])**.** *Let g be an involution of Co_1 and $\tilde{g}$ a lift of g in Co_0.*

1. *If g is of class $2A$, then $\tilde{g} = \varepsilon_{\mathcal{O}}^{\mathcal{F}}$ or $\varepsilon_{\Omega+\mathcal{O}}^{\mathcal{F}}$ for some octad $\mathcal{O}$ and an 8-frame $\mathcal{F}$.*
2. *If g is of class $2B$, then $\tilde{g}$ is of order 4.*
3. *If g is of class $2C$, then $\tilde{g} = \varepsilon_{\mathcal{S}}^{\mathcal{F}}$ for some dodecad $\mathcal{S}$ and an 8-frame $\mathcal{F}$.*

Now, we shall consider the embeddings of the lattice $\sqrt{2}E_8$ into the Leech lattice Λ. First we shall construct some sublattices isomorphic to $\sqrt{2}E_8$ in Λ.

Let $\mathcal{F} = \{\pm 2\alpha_1, \dots, \pm 2\alpha_{24}\}$ be an 8-frame of Λ. Then $\{\alpha_1, \alpha_2, \dots, \alpha_{24}\}$ is an orthogonal basis of $\mathbb{R}^{24}$ such that $\langle \alpha_i, \alpha_j\rangle = 2\delta_{i,j}$.

Lemma 4.6. *Let $\mathcal{O} \in \mathcal{C}_{\mathcal{F}}$ be an octad and denote*

$$E_{\mathcal{F}}(\mathcal{O}) = \mathrm{span}_{\mathbb{Z}} \left\{ \alpha_i \pm \alpha_j, \frac{1}{2}\sum_{i\in\mathcal{O}} \epsilon_i \alpha_i \right\},$$

where $i, j \in \mathcal{O}$ and $\epsilon_i = \pm 1$ such that $\prod_{i\in\mathcal{O}} \epsilon_i = 1$. Then $E_{\mathcal{F}}(\mathcal{O})$ is isomorphic to $\sqrt{2}E_8$.

Theorem 4.7 ([10])**.** *The Conway group Co_0 is transitive on the set $\{E \subset \Lambda \mid E \cong \sqrt{2}E_8\}$ of sublattices of the Leech lattice isomorphic to $\sqrt{2}E_8$.*

As a corollary, we have the following proposition.

Proposition 4.8. *Let $E \subset \Lambda$ be a sublattice isomorphic to $\sqrt{2}E_8$. Then there exists an 8-frame $\mathcal{F}$ of Λ and an octad $\mathcal{O} \in \mathcal{C}_{\mathcal{F}}$ such that $E_{\mathcal{F}}(\mathcal{O}) = E$.*

Proof. Fix an 8-frame $\mathcal{F}_0 = \{\pm\beta_1, \cdots, \pm\beta_{24}\}$ and an octad $\mathcal{O} \in \mathcal{C}_{\mathcal{F}_0}$. For any $E \cong \sqrt{2}E_8$, there is $g \in \operatorname{Aut}\Lambda$ such that $g(E) = E_{\mathcal{F}_0}(\mathcal{O})$. Then $\mathcal{F} = \{\pm g^{-1}(\beta_1), \ldots, \pm g^{-1}(\beta_{24})\}$ forms an 8-frame of Λ and E is clearly equal to $E_{\mathcal{F}}(\mathcal{O})$, where we regard $\mathcal{O}$ as an element in $\mathcal{C}_{\mathcal{F}}$. □

Lemma 4.9. (cf. [3, 10]) *For each $E \cong \sqrt{2}E_8 \subset \Lambda$, there exists 135 distinct 8-frames $\mathcal{F}$ of Λ such that $E = E_{\mathcal{F}}(\mathcal{O})$.*

Proof. Let $\mathcal{F}_0$ be an 8-frame of E. Then $\mathcal{F}_0$ forms a coset of $E/2E$ and, $\mathcal{F}_0$ is contained in a coset of $\Lambda/2\Lambda$. This implies that there exists a unique 8-frame $\mathcal{F}$ of Λ such that $\mathcal{F}_0 \subset \mathcal{F}$ (cf. [3, 10]). It follows from Remark 3.6 that E contains exactly 135 distinct 8-frames, which proves this lemma. □

4.2. **Automorphism group of V_Λ^+.** The full automorphism group of V_Λ^+ has been determined in [15]. We shall recall its main results.

Let L be a positive definite even lattice and

$$1 \longrightarrow \langle\kappa\rangle \longrightarrow \hat{L} \xrightarrow{-} L \longrightarrow 1$$

a central extension of L by $\langle\kappa\rangle \cong \mathbb{Z}_2$ such that the commutator map $c_0(\alpha,\beta) = \langle\alpha,\beta\rangle \bmod 2$, $\alpha,\beta \in L$. The following theorem is well-known (cf. [7]):

Theorem 4.10. *For an even lattice L, the sequence*

$$1 \longrightarrow \operatorname{Hom}(L,\mathbb{Z}_2) \longrightarrow \operatorname{Aut}\hat{L} \longrightarrow \operatorname{Aut}L \longrightarrow 1$$

is exact. In particular $\operatorname{Aut}\hat{\Lambda} \cong 2^{24}\cdot Co_0$.

Recall that θ is the automorphism of V_Λ defined by

$$\theta(\alpha_1(-n_1)\cdots\alpha_k(-n_k)e^\alpha) = (-1)^k\alpha_1(-n_1)\cdots\alpha_k(-n_k)\theta(e^\alpha),$$

where $\theta(a) = a^{-1}\kappa^{\langle\bar{a},\bar{a}\rangle/2}$ on $\hat{L}$.

Lemma 4.11. *([15]) Let L be a positive definite even lattice without roots, i.e., $L(1) = \emptyset$. Then the centralizer $C_{\operatorname{Aut}V_L}(\theta)$ of θ in $\operatorname{Aut}V_L$ is isomorphic to $\operatorname{Aut}\hat{L}$. If $L = \Lambda$ is the Leech lattice, we have*

$$C_{\operatorname{Aut}V_\Lambda}(\theta) \cong \operatorname{Aut}\hat{\Lambda} \cong 2^{24}\cdot Co_0.$$

Proposition 4.12. *Let $E = E_{\mathcal{F}}(\mathcal{O}) \cong \sqrt{2}E_8$ be a sublattice of Λ and e an Ising vector in V_Λ of the form $\varphi_x(e_E)$. Then $\tau_e \in C_{\mathrm{Aut}\,V_\Lambda}(\theta)$ and the image of τ_e under the canonical epimorphism from $C_{\mathrm{Aut}\,V_\Lambda}(\theta)(\cong \mathrm{Aut}\,\hat{\Lambda})$ to $C_{\mathrm{Aut}\,V_\Lambda}(\theta)/\mathrm{Hom}(\Lambda,\mathbb{Z}_2) \cong \mathrm{Aut}\,\Lambda = Co_0$ is $\varepsilon_{\mathcal{O}}^{\mathcal{F}}$.*

Proof. Since e is fixed by θ, it is clear that $\tau_e \in C_{\mathrm{Aut}\,V_\Lambda}(\theta)$.

Let g be the image of τ_e under the canonical epimorphism from $\mathrm{Aut}\,\hat{\Lambda}$ to $\mathrm{Aut}\,\Lambda$. Let $B = \{\alpha \in \Lambda \mid \langle \alpha, \beta\rangle = 0 \text{ for all } \beta \in E\}$ be the orthogonal complement of E in Λ. Then B is isomorphic to the Barnes-Wall lattice BW_{16} (cf. [3, 10]). By the definition of e, $e_1 v = 0$ for all $v \in V_B$. Hence $\tau_{e|_{V_B}} = \mathrm{id}$ but τ_e acts as θ on V_E by Theorem 3.5. Thus g acts as 1 on B and -1 on E. As $E = E_{\mathcal{F}}(\mathcal{O})$, we have $g = \varepsilon_{\mathcal{O}}^{\mathcal{F}}$ as desired. □

Theorem 4.13. *([15]) Let $V_\Lambda^+ = \{v \in V_\Lambda \mid \theta(v) = v\}$ be the fixed point subVOA of θ in V_Λ. Then $\mathrm{Aut}\,V_\Lambda^+ \cong C_{\mathrm{Aut}\,V_\Lambda}(\theta)/\langle\theta\rangle \cong 2^{24}\cdot Co_1$ and the sequence*

$$1 \longrightarrow \mathrm{Hom}(\Lambda,\mathbb{Z}_2) \longrightarrow \mathrm{Aut}\,V_\Lambda^+ \stackrel{\xi}{\longrightarrow} \mathrm{Aut}\,\Lambda/\langle \pm 1\rangle \longrightarrow 1.$$

is exact.

5. Ising Vectors in the $\mathbb{Z}_2$-orbifold VOA V_Λ^+

In this section, we shall describe the Ising vectors in V_Λ^+ and discuss the corresponding symmetries of the moonshine VOA $V^\natural$.

5.1. **Characterization of Ising vectors in V_Λ^+.** In [11], all Ising vectors in the VOA $V_{\sqrt{2}R}^+$ has been characterized for a root lattice R as follows.

Proposition 5.1. *([11]) Let R be a root lattice. Then any Ising vector in $V_{\sqrt{2}R}^+$ is either equal to $\omega^\pm(\alpha)$ for some $\alpha \in \sqrt{2}R$ of norm 4 or $\varphi_x(e_E)$ for some sublattice E of $\sqrt{2}R$ isomorphic to $\sqrt{2}E_8$ and $x \in \frac{1}{2}E$.*

Note that $\omega^\pm(\alpha) \in V_{\mathbb{Z}\alpha}^+ \cong V_{\sqrt{2}A_1}^+$ and that $\varphi_x(e_E) \in V_E^+ \cong V_{\sqrt{2}E_8}^+$. Its generalization was given in [11] as a conjecture.

Conjecture 5.2. *Let L be an even lattice L without roots. Then any Ising vector of V_L^+ belongs to a subVOA V_U^+ for some sublattice U of L isomorphic to $\sqrt{2}A_1$ or $\sqrt{2}E_8$.*

We shall show that the conjecture holds when L is isomorphic to the Leech lattice Λ. First let us recall a result from [11].

Lemma 5.3. [11, Lemma 3.7] *Let $V = \oplus_{i=0}^{\infty} V_n$ be a VOA with $V_0 = \mathbb{C}\mathbf{1}$, $V_1 = 0$. Suppose that V has two Ising vectors e, f and that e is of σ-type, i.e., $\tau_e = \mathrm{id}$ on V. Then e is fixed by τ_f, namely $e \in V^{\tau_f}$.*

Let us discuss Ising vectors of V_L^+ and the associated Miyamoto involutions.

Lemma 5.4. *Let L be an even lattice of rank n without roots. Let U be the sublattice of L generated by $L(2)$. Suppose that $U \cong \sqrt{2}R$ for some root lattice R of rank m. Then any Ising vector of V_L^+ belongs to V_U^+.*

Proof. Let $U^\perp = \{\alpha \in L \mid \langle \alpha, \beta\rangle = 0 \text{ for all } \beta \in U\}$ be the orthogonal complement of U in L. Let $\mathfrak{h}_0 = \mathbb{C}\otimes_{\mathbb{Z}} U$ and $\mathfrak{h}_1 = \mathbb{C}\otimes_{\mathbb{Z}} U^\perp$. Then $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{Z}} L = \mathfrak{h}_0 \perp \mathfrak{h}_1$. Since the weight 2 subspace of $(V_L^+)_2$ is equal to the weight 2 subspace of $[S(\hat{\mathfrak{h}}^-)\otimes \mathbb{C}\{U\}]^+ \cong [V_U \otimes S(\hat{\mathfrak{h}}_1^-)]^+$, we know that any Ising vector e of V_L^+ must be in $[V_U \otimes S(\hat{\mathfrak{h}}_1^-)]^+$.

Let $U' \cong (\sqrt{2}A_1)^{m-n}$ and $K = U \perp U'$. Then both L and K are of rank n and we can identify $\mathbb{C}\otimes_{\mathbb{Z}} K$ with $\mathfrak{h}$ and $\mathbb{C}\otimes_{\mathbb{Z}} U'$ with $\mathfrak{h}_1$. Then the lattice VOA V_K is given by

$$V_K = S(\hat{\mathfrak{h}}^-)\otimes \mathbb{C}\{K\} \cong S(\hat{\mathfrak{h}}_0^-)\otimes S(\hat{\mathfrak{h}}_1^-)\otimes \mathbb{C}\{U\}\otimes \mathbb{C}\{U'\},$$

which contains $V_U \otimes S(\hat{\mathfrak{h}}_1^-)$ as a subalgebra. Hence, we can view the Griess algebra of V_L^+ as a Griess subalgebra of V_K^+, also. Thus by Proposition 5.1, $e \in V_U^+$ or $e \in V_{U'}^+$ as $K/\sqrt{2}$ is isomorphic to a root lattice. It follows that $e \in V_U^+$ as there is no Ising vector in $S(\hat{\mathfrak{h}}_1^-)$. □

Lemma 5.5. *Let L be an even lattice without roots and e an Ising vector of V_L^+. Then the automorphism τ_e belongs to $C_{\mathrm{Aut}\,V_L}(\theta)/\langle\theta\rangle$. In particular, $\tau_e \in \mathrm{Aut}\,\hat{L}/\langle\theta\rangle$.*

Proof. We view τ_e as an automorphism of V_L. Since θ fixes e, we have $\theta\tau_e\theta = \tau_{\theta(e)} = \tau_e$, which proves this lemma. □

Since $C_{\mathrm{Aut}\,V_\Lambda}(\theta)/\langle\theta\rangle \cong 2^{24}\cdot Co_1$, it suffices to consider the two cases: $\xi(\tau_e) = 1$ and $\xi(\tau_e) \neq 1$, where $\xi : C_{\mathrm{Aut}\,V_\Lambda}(\theta)/\langle\theta\rangle \to Co_1$ denotes the natural epimorphism. First, we study the case $\xi(\tau_e) = 1$ for general L.

Proposition 5.6. *Let L be an even lattice without roots and let e be an Ising vector of V_L^+. Suppose that $\xi(\tau_e)=1$. Then e is equal to $\omega^{\pm}(\alpha)$ for some $\alpha\in L(2)$ or $\varphi_x(e_E)$ for some $E\subset L$ isomorphic to $\sqrt{2}E_8$ and $x\in\frac{1}{2}E$.*

Proof. By Lemma 5.5, $\tau_e\in C_{\mathrm{Aut}\,V_L}(\theta)/\langle\theta\rangle$. By the assumption, $\tau_e=\varphi_v$ for some $v\in L^*/2L^*$. Then $e\in(V_L^+)^{\varphi_v}$ and by Proposition 3.2 $(V_L^+)^{\varphi_v}=V_{L_v}^+$, where $L_v=\{u\in L\mid\langle v,u\rangle\in 2\mathbb{Z}\}$. Note that e is of σ-type in $V_{L_v}^+$. Set $A=\langle\tau_{\omega^{\pm}(\alpha)}\mid\alpha\in L_v(2)\rangle$. By Lemma 5.3, $e\in(V_{L_v}^+)^A$. Moreover $(V_{L_v}^+)^A=V_{\tilde{L}_v}^+$, where $\tilde{L}_v=\{u\in L_v\mid\langle u,w\rangle\in 2\mathbb{Z},\ \forall w\in L_v(2)\}$. Let U be the sublattice generated by $\tilde{L}_v(2)$. In this case, $(1/\sqrt{2})U$ is isomorphic to a root lattice. Thus, by Lemma 5.4, $e\in V_U^+$ and $e=\omega^{\pm}(\alpha)$ for some $\alpha\in\tilde{L}_v(2)$ or $e=\varphi_x(e_E)$ by Proposition 5.1. □

Next, we consider the case $\xi(\tau_e)\neq 1$ for general L.

Lemma 5.7. *Let L be an even lattice without roots. Let e be an Ising vector in V_L^+ such that $\xi(\tau_e)\neq 1$ and let $g\in\mathrm{Aut}\,L$ be a lift of $\xi(\tau_e)$. Then $e\in V_{L\cap 2N(e)^*}^+$, where $N(e)$ is the sublattice generated by $\{v\in L(2)\mid g(v)=\pm v\}$.*

Proof. Set $U^i=\{v\in L\mid g(v)=(-1)^iv\}$ $(i=0,1)$ and $U=U^0\oplus U^1$. Since for $\lambda\in L$ both $\lambda+g(\lambda)$ and $\lambda-g(\lambda)$ belong to U, we have $2L\subset U$. Consider the coset $L/U\subset((U^0)^\perp\oplus(U^1)^\perp)/(U^0\oplus U^1)$ and the canonical projections $\lambda\in L/U$ to $\lambda_i\in(U^i)^\perp/U^i$ for $i=0.1$. Then we obtain the decomposition

$$(V_L^+)^{\tau_e}=\bigoplus_{\lambda_0+\lambda_1\in L/U}V_{\lambda_0+U_0}^{\tau_e\theta}\otimes V_{\lambda_1+U_1}^{\tau_e},$$

where $V_{\lambda_0+U_0}^{\tau_e\theta}$ is the $\tau_e\theta$-fixed point subspace of $V_{\lambda_0+U_0}$ and $V_{\lambda_1+U_1}^{\tau_e}$ is the τ_e-fixed point subspace of $V_{\lambda_1+U_1}$. Set $A=\langle\tau_{\omega^{\pm}(\alpha)}\mid\alpha\in U(2)\rangle$. Then $N(e)$ is a sublattice of L generated by $U(2)$ and $A=\langle\varphi_\alpha\mid\alpha\in N(e)\rangle$. Since e is of σ-type in $(V_L^+)^{\tau_e}$, $e\in(V_L^+)^{\langle\tau_e,A\rangle}=(V_{L\cap 2N(e)^*}^+)^{\tau_e}\subset V_{L\cap 2N(e)^*}^+$ by Lemma 5.3. □

We apply the lemma above to characterize Ising vectors of V_Λ^+. Let e be an Ising vector in V_Λ^+ such that $\xi(\tau_e)\neq 1$. Then by Proposition 4.5, the conjugacy class of $\xi(\tau_e)\in Co_1$ is $2A$ or $2C$ since τ_e is an involution in $\mathrm{Aut}\,V_\Lambda$. Hence, we should consider these two cases.

Proposition 5.8. *Let e be an Ising vector in V_Λ^+. Suppose that $\xi(\tau_e)$ belongs to the conjugacy class $2A$ of Co_1. Then $N(e) = E \oplus E'$ and $e = \varphi_x(e_E)$, where $E \cong \sqrt{2}E_8$, $E' \cong \mathrm{BW}_{16}$ and $x \in \frac{1}{2}E$.*

Proof. By Proposition 4.5, $\xi(\tau_e) = \varepsilon_{\mathcal{O}}^{\mathcal{F}}$ or $\varepsilon_{\Omega+\mathcal{O}}^{\mathcal{F}}$ for some octad $\mathcal{O}$. Thus, by Theorem 4.4, $N(e) = E \oplus E'$, where $E \cong \sqrt{2}E_8$ and $E' \cong \mathrm{BW}_{16}$. Then $\Lambda \cap 2N(e)^* \cong \sqrt{2}E_8 \oplus 2\mathrm{BW}_{16}^*$. By Lemma 5.7, $e \in V^+_{\sqrt{2}E_8 \oplus 2\mathrm{BW}_{16}^*}$. Since $2\mathrm{BW}_{16}^* \cong \sqrt{2}\mathrm{BW}_{16}$, we have $2\mathrm{BW}_{16}^*(2) = \emptyset$. Then by Lemma 5.4, $e \in V_E^+$. Hence by Proposition 5.1, $e = \omega^\pm(\alpha)$ or $\varphi_x(e_E)$. By Proposition 3.2, e must be $\varphi_x(e_E)$ for some $x \in \frac{1}{2}E$ as $\xi(\tau_e) \neq 1$. $\square$

Proposition 5.9. *There is no Ising vector e in V_Λ^+ such that $\xi(\tau_e)$ belongs to the conjugacy class $2C$ of Co_1.*

Proof. Let e be an Ising vector in V_Λ^+ such that $\xi(\tau_e)$ belongs to the conjugacy class $2C$ of Co_1. By Theorem 4.4 and Proposition 4.5, $N(e) \cong \sqrt{2}D_{12} \oplus \sqrt{2}D_{12}$. Then $\Lambda \cap 2N(e)^* \cong \sqrt{2}D_{24}^+$. By Lemma 5.7, $e \in V^+_{\sqrt{2}D_{24}^+}$. Since the set of norm 4 vectors in $\sqrt{2}D_{24}^+$ generates $\sqrt{2}D_{24}$, e belongs to $V^+_{\sqrt{2}D_{24}}$. Note that there is no sublattice of $\sqrt{2}D_{24}$ isomorphic to $\sqrt{2}E_8$. Hence, by Proposition 5.1, $e = \omega^\pm(\alpha)$ for some $\alpha \in \sqrt{2}D_{24}(2)$. In this case, $\xi(\tau_e) = 1$ by Proposition 3.2, which is a contradiction. $\square$

Therefore, the conjecture holds when L is isomorphic to the Leech lattice. In particular, there are exactly 2 types of Ising vectors in V_Λ^+, namely, $\omega^\pm(\alpha)$ for some $\alpha \in \Lambda(2)$ or $\varphi_x(e_E)$ for some sublattice $E \cong \sqrt{2}E_8$ and $\frac{1}{2}x \in E$

Since Co_0 is transitive on $\Lambda(2)$, for any $\alpha \in \Lambda(2)$, there is a sublattice $E \cong \sqrt{2}E_8$ such that $\alpha \in E$. Therefore we also have

Theorem 5.10. *Let Λ be the Leech lattice. Then for any Ising vector e in V_Λ^+, there is a sublattice $E \cong \sqrt{2}E_8$ such that $e \in V_E^+$.*

As a corollary, we can count all Ising vectors in V_Λ^+.

Corollary 5.11. *There are exactly* 11935319760 *Ising vectors in the VOA V_Λ^+.*

Proof. Since Λ has exactly 196560 vectors of squared norm 4, there are 196560 Ising vectors of the form $\omega^\pm(\alpha), \alpha \in \Lambda(2)$. By Proposition 4.8, any sublattice isomorphic to

$\sqrt{2}E_8$ is determined by an 8-frame of Λ and an octad. By Lemma 4.9, there are exactly

$$|\{\mathcal{O}\in\mathcal{C}|\ |\mathcal{O}|=8\}|\times\frac{|\Lambda(4)|}{48}\times\frac{1}{135}=759\times\frac{398034000}{48}\times\frac{1}{135}=46621575$$

sublattices isomorphic to $\sqrt{2}E_8$. Thus, there are $11935123200(=46621575\times 256)$ Ising vectors of the form $\varphi_x(e_E)$. Therefore, there are totally 11935319760 Ising vectors in V_Λ^+. □

Corollary 5.12. *None of the Ising vectors in V_Λ^+ is of σ-type, namely $V_e(\frac{1}{16})\neq 0$ for any Ising vector $e\in V_\Lambda^+$.*

Proof. Let e be an Ising vector in V_Λ^+. Then by Theorem 5.10, $e=\omega^\pm(\alpha)$ or $e=\varphi_x(e_E)$, and τ_e is not of σ-type by Proposition 3.2 and 4.12 respectively. □

Let us consider the correspondence between Ising vectors in V_Λ^+ and Miyamoto automorphisms.

Proposition 5.13. *Let e,f be Ising vectors in V_Λ^+ such that $\tau_e=\tau_f\in\operatorname{Aut}V_\Lambda^+$.*

(1) *If $e=\omega^\pm(\alpha)$ then f is equal to $\omega^+(\alpha)$ or $\omega^-(\alpha)$.*
(2) *If $e=\varphi_x(e_E)$ for some sublattice $E\cong\sqrt{2}E_8$ of Λ and $x\in\frac{1}{2}E$ then $f=e$.*

Proof. First we suppose that $e=\omega^\pm(\alpha)$. Then by Proposition 3.2 and 4.12, $f=\omega^\pm(\beta)$ for some $\beta\in\Lambda(2)$, and $\langle\beta,v\rangle=\langle\alpha,v\rangle \mod 2$ for all $v\in\Lambda$. Since Λ is unimodular, $\beta\in\alpha+2\Lambda$, and (1) follows.

Next we suppose that $e=\varphi_x(e_E)$. By Proposition 3.2 and 4.12, f must be $\varphi_y(e_E)$ for some $y\in\frac{1}{2}E$. Then

$$\tau_e\tau_f=\varphi_{2(x+y)}=\mathrm{id}$$

since $\tau_{\varphi_v(e_E)}=\varphi_{2v}\tau_{e_E}$ for $v\in\frac{1}{2}E$. Recall that if $u\in E$ satisfies $\langle u,\Lambda\rangle\in 2\mathbb{Z}$ then $u\in 2E$. Hence $2(x+y)\in 2E$, namely $y\in x+E$, and we obtain $e=f$. □

At end of this subsection, we give some applications to Ising vectors in the moonshine VOA $V^\natural=V_\Lambda^+\oplus V_\Lambda^{T,+}$. Let z be the linear map of $V^\natural$ acting as 1 and -1 on V_Λ^+ and $V_\Lambda^{T,+}$ respectively. Then z is an automorphism of $V^\natural$.

Lemma 5.14. *Let $\alpha\in\Lambda(2)$. Then $\tau_{\omega^+(\alpha)}\tau_{\omega^-(\alpha)}=z$ on $V^\natural$.*

Proof. Set $g = \tau_{\omega^+(\alpha)}\tau_{\omega^-(\alpha)}$. By Proposition 3.2, $g = 1$ on V_Λ^+. Since the V_Λ^+-module $V_\Lambda^{T,+}$ is irreducible, g must be 1 or -1 on $V_\Lambda^{T,+}$.

Suppose that $g = 1$ on $V_\Lambda^{T,+}$. Let $F = \{\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_{24}\}$ be a 4-frame of Λ containing α (cf. Lemma 4.3) and $\mathcal{T} = \{\omega^\pm(\alpha_1), \ldots, \omega^\pm(\alpha_{24})\}$ the Virasoro frame of $V^\natural$ associated with F. Let (C, D) be the structure code of $V^\natural$ with respect to $\mathcal{T}$, namely C and D are the binary codes of length 48 such that V^0 is a code VOA associated with C and $V^\natural = \oplus_{d\in D} V^d$, where V^d is an irreducible V^0-module whose $\frac{1}{16}$-word is $d \in D$ (cf. [4, 14]). By the assumption, for any codeword $d \in D$, its components corresponding to $\omega^+(\alpha)$ and $\omega^-(\alpha)$ are the same. Therefore, the binary word $(1100\cdots 00) \in D^\perp$. Since $V^\natural$ is holomorphic, $D^\perp = C$. Hence C contains a codeword of weight 2, which contradicts $V_1^\natural = 0$. Hence $g = -1$ on $V_\Lambda^{T,+}$, which completes the lemma. □

Proposition 5.15. *None of the Ising vectors in $V^\natural$ is of σ-type, namely $V_e(\frac{1}{16}) \neq 0$ for any Ising vector $e \in V^\natural$.*

Proof. Let $\alpha \in \Lambda(2)$. Then $\tau_{\omega^+(\alpha)}\tau_{\omega^-(\alpha)} = z$ by the above lemma. Now suppose that e is of σ-type. By Lemma 5.3, $e \in (V^\natural)^{\langle \tau_{\omega^+(\alpha)}, \tau_{\omega^-(\alpha)}\rangle} \subset (V^\natural)^z = V_\Lambda^+$ and hence e is of σ-type in V_Λ^+, which contradicts Corollary 5.12. □

Proposition 5.13 and Lemma 5.14 show that the map $e \mapsto \tau_e$ from Ising vectors in $V_\Lambda^+ \subset V^\natural$ to $\operatorname{Aut} V^\natural$ is injective.

Proposition 5.16. *Let e, f be Ising vectors in V_Λ^+ such that $\tau_e = \tau_f$ on $V^\natural$. Then $e = f$.*

5.2. **The centralizer of τ_e in $C_{\operatorname{Aut} V^\natural}(z)$.** In this subsection, we shall determine the centralizer of τ_e in $C_{\operatorname{Aut} V^\natural}(z)$ for any Ising vector e in V_Λ^+. Note that there are exactly 2 types of Ising vectors in V_Λ^+, namely, $\omega^\pm(\alpha)$ for some $\alpha \in \Lambda(2)$ or $\varphi_x(e_E)$ for some sublattice $E \cong \sqrt{2}E_8$ and $x \in \frac{1}{2}E$.

By a group of type 2^{n+m}, we mean a group H for which there exists an exact sequence

$$1 \to \mathbb{Z}_2^n \to H \to \mathbb{Z}_2^m \to 1$$

which does not necessarily split. A group G of type $2^{n_1+\cdots+n_k}$ can be defined inductively by the exact sequence

$$1 \to 2^{n_1+\cdots+n_{k-1}} \to G \to 2^{n_k} \to 1$$

First, we consider Ising vectors in V_Λ^+ associated with norm 4 vectors in Λ.

Lemma 5.17. *The centralizer of z in $\mathrm{Aut}\,V^\natural$ has the structure $2^{1+24}.Co_1$.*

Proof. Note that $V^\natural = V_\Lambda^+ \oplus V_\Lambda^{T,+}$ is a simple current extension[1] of V_Λ^+ and that for any automorphism $g \in \mathrm{Aut}\,V_\Lambda^+$, the g-conjugate module $g\circ V_\Lambda^{T,+}$, i.e, $g\circ V_\Lambda^{T,+} = V_\Lambda^{T,+}$ as vector spaces and the vertex operator Y_g is defined by $Y_g(u,z) = Y(g^{-1}u,z)$, is isomorphic to $V_\Lambda^{T,+}$ itself. In other word, $V_\Lambda^{T,+}$ is preserved by $\mathrm{Aut}\,V_\Lambda^+$. Hence the following sequence is exact (cf. [15]):

$$1 \to \langle z\rangle \to C_{\mathrm{Aut}\,V^\natural}(z) \xrightarrow{\mu} \mathrm{Aut}\,V_\Lambda^+ \to 1.$$

□

Theorem 5.18. *Let $\alpha \in \Lambda(2)$ and $\varepsilon \in \{\pm\}$. Set $e = \omega^\varepsilon(\alpha)$. Then there is an exact sequence*

$$1 \to \{\varphi_\beta \in \mathrm{Hom}(\Lambda,\mathbb{Z}_2) \mid \langle\beta,\alpha\rangle \in 2\mathbb{Z}\} \to C_{\mathrm{Aut}\,V^\natural}(z,\tau_e)/\langle z\rangle \to Co_2 \to 1.$$

In particular, the centralizer $C_{\mathrm{Aut}\,V^\natural}(\tau_e,z)$ has the structure $2^{2+22}.Co_2$.

Proof. By Proposition 5.17, $C_{\mathrm{Aut}\,V^\natural}(z)/\langle z\rangle \cong \mathrm{Aut}\,V_\Lambda^+$. It follows from Proposition 5.16 that $C_{\mathrm{Aut}\,V^\natural}(z,\tau_e)/\langle z\rangle$ is isomorphic to the stabilizer $\mathrm{Stab}_{\mathrm{Aut}\,V_\Lambda^+}(e)$ of e in $\mathrm{Aut}\,V_\Lambda^+$.

We view τ_e as an automorphism of V_Λ^+ and consider the exact sequence in Theorem 4.10:

$$1 \longrightarrow \mathrm{Hom}(\Lambda,\mathbb{Z}_2) \longrightarrow \mathrm{Aut}\,V_\Lambda^+ \xrightarrow{\ \xi\ } \mathrm{Aut}\,\Lambda/\langle\pm1\rangle \longrightarrow 1.$$

It follows from Proposition 3.2 that $\tau_e = \varphi_\alpha \in \mathrm{Hom}(\Lambda,\mathbb{Z}_2)$ for some $\alpha\in\Lambda$.

For $\varphi_\beta \in \mathrm{Hom}(\Lambda,\mathbb{Z}_2)$, $\beta\in\Lambda$, it is easy to see that $\varphi_\beta(\omega^\pm(\alpha)) = \omega^\pm(\alpha)$ if $\langle\alpha,\beta\rangle \in 2\mathbb{Z}$, and $\varphi_\beta(\omega^\pm(\alpha)) = \omega^\mp(\alpha)$ if $\langle\alpha,\beta\rangle\in 2\mathbb{Z}+1$. Hence $\mathrm{Hom}(\Lambda,\mathbb{Z}_2)\cap \mathrm{Stab}_{\mathrm{Aut}\,V_\Lambda^+}(e) = \{\varphi_\beta \in \mathrm{Hom}(\Lambda,\mathbb{Z}_2) \mid \langle\alpha,\beta\rangle \in 2\mathbb{Z}\} \cong 2^{23}$.

Let g be an element in $\mathrm{Aut}\,V_\Lambda^+$ fixing e. Then $g(\varphi_\alpha) = \varphi_{\xi(g)(\alpha)}$, hence $\xi(g)\in \mathrm{Stab}_{Co_1}(\alpha+2\Lambda) = Co_2$. Conversely, let $p\in \mathrm{Aut}\,V_\Lambda^+$ be a lift of an element in Co_2. Then $p(e) = \omega^+(\alpha)$

[1] An irreducible module M of a simple VOA U is said to be a simple current module if the fusion product $M\times_U W$ is also irreducible for all irreducible U-module W. A VOA V is a simple current extension of a subVOA U if V is a direct sum of simple current modules of U.

or $\omega^-(\alpha)$. If $p(e) \neq e$ then $p \circ \varphi_\gamma(e) = e$, where $\gamma \in \Lambda$ such that $\langle \gamma, \alpha\rangle \in 2\mathbb{Z}+1$. This shows that $\xi(\mathrm{Stab}_{\mathrm{Aut}\, V_\Lambda^+}(e)) = Co_2$.

Thus we obtain an exact sequence

$$1 \to \{\varphi_\beta \in \mathrm{Hom}(\Lambda, \mathbb{Z}_2) \mid \langle \beta, \alpha\rangle \in 2\mathbb{Z}\} \to \mathrm{Stab}_{\mathrm{Aut}\, V_\Lambda^+}(e) \to Co_2 \to 1.$$

Since $\{\varphi_\beta \in \mathrm{Hom}(\Lambda, \mathbb{Z}_2) \mid \langle \beta, \alpha\rangle \in 2\mathbb{Z}\} = \langle \tau_{\omega^+(\beta)} \mid \langle \beta, \alpha\rangle \in 2\mathbb{Z}\rangle$ in $\mathrm{Aut}\, V_\Lambda^+$, we have

$$\begin{aligned}\mu^{-1}(\{\varphi_\beta \in \mathrm{Hom}(\Lambda, \mathbb{Z}_2) \mid \langle \beta, \alpha\rangle \in 2\mathbb{Z}\}) &= \langle z, \tau_{\omega^+(\beta)} \mid \langle \beta, \alpha\rangle \in 2\mathbb{Z}\rangle \\ &\subset O_2(C_{\mathrm{Aut}\, V^\natural}(z)) \cong 2_+^{1+24}\end{aligned}$$

in $\mathrm{Aut}\, V^\natural$, where μ is the natural epimorphism $C_{\mathrm{Aut}\, V^\natural}(z) \to C_{\mathrm{Aut}\, V^\natural}(z)/\langle z\rangle \cong \mathrm{Aut}\, V_\Lambda^+$.

It is clear that z and τ_e are in the center of $C_{\mathrm{Aut}\, V^\natural}(\tau_e, z)$. Hence $\mu^{-1}(\{\varphi_\beta \in \mathrm{Hom}(\Lambda, \mathbb{Z}_2) \mid \langle \beta, \alpha\rangle \in 2\mathbb{Z}\})$ is of shape 2^{2+22}, which proves this theorem. □

If $e = \varphi_x(e_E)$, it comes out that the centralizer of τ_e in $C_{\mathrm{Aut}\, V^\natural}(z)$ also stabilizes the VOA $V_E^+ \subset V_\Lambda^+$.

Proposition 5.19. *Let E be a sublattice of Λ isomorphic to $\sqrt{2}E_8$ and let $x \in \frac{1}{2}E$. Set $e = \varphi_x(e_E)$. Then $C_{\mathrm{Aut}\, V_\Lambda^+}(\tau_e)$ stabilizes V_E^+. Moreover, $C_{\mathrm{Aut}\, V^\natural}(\tau_e, z)$ stabilizes V_E^+.*

Proof. Let $\mathcal{F}$ and $\mathcal{O}$ be an 8-frame and an octad such that $E = E_{\mathcal{F}}(\mathcal{O})$. Then by Proposition 4.12, we obtain the exact sequence

$$1 \to \{f_\alpha \in \mathrm{Hom}(\Lambda, \mathbb{Z}_2) \mid \langle \alpha, E\rangle \in 2\mathbb{Z}\} \to C_{\mathrm{Aut}\, V_\Lambda^+}(\tau_e) \to C_{Co_1}(\varepsilon_{\mathcal{O}}^{\mathcal{F}}) \to 1.$$

Since $C_{Co_1}(\varepsilon_{\mathcal{O}}^{\mathcal{F}})$ preserves E, $C_{\mathrm{Aut}\, V_\Lambda^+}(\tau_e)$ stabilizes V_E^+. It follows from Lemma 5.17 and $z = \mathrm{id}$ on V_Λ^+ that $C_{\mathrm{Aut}\, V^\natural}(\tau_e, z)$ preserves V_E^+. □

In order to determine the centralizer $C_{\mathrm{Aut}\, V^\natural}(\tau_e, z)$ of τ_e and z in $\mathrm{Aut}\, V^\natural$, we need the following lemma.

Lemma 5.20. *Let E be a lattice isomorphic to $\sqrt{2}E_8$. Then the stabilizer of e_E in the subgroup $\mathrm{Aut}\, \hat{E}/\langle\theta\rangle$ of $\mathrm{Aut}\, V_E^+$ is isomorphic to $\mathrm{Aut}\, E/\langle -1\rangle \cong O^+(8,2)$.*

Proof. Since $\langle E, E\rangle \subset 2\mathbb{Z}$, $\mathrm{Aut}\, \hat{E}/\langle\theta\rangle$ is a split extension of $\mathrm{Aut}\, E/\langle -1\rangle$ by $\mathrm{Hom}(E, \mathbb{Z}_2)$ (cf. [7]). Clearly for $\varphi_\alpha \in \mathrm{Hom}(E, \mathbb{Z}_2)$, $\varphi_\alpha(e_E) = e_E$ if and only if $\varphi_\alpha = 1$. By the definition of e_E, the subgroup $\mathrm{Aut}\, E/\langle -1\rangle$ fixes e_E. □

Theorem 5.21. *Let E be a sublattice of Λ isomorphic to $\sqrt{2}E_8$ and x a vector in $\frac{1}{2}E$. Set $e = \varphi_x(e_E)$. Then the centralizer $C_{\mathrm{Aut}\,V^\natural}(\tau_e, z)$ has the structure $2^{2+8+16}.\Omega^+(8,2)$.*

Proof. By Proposition 5.19, $C_{\mathrm{Aut}\,V^\natural}(\tau_e, z)/\langle z\rangle$ is a subgroup of the stabilizer $\mathrm{Stab}_{\mathrm{Aut}\,V_\Lambda^+}(V_E^+)$ of V_E^+ in $\mathrm{Aut}\,V_\Lambda^+$.

Let us describe $\mathrm{Stab}_{\mathrm{Aut}\,V_\Lambda^+}(V_E^+)$. It follows from Proposition 3.2 that $\mathrm{Hom}(\Lambda,\mathbb{Z}_2)$ is a subgroup of $\mathrm{Stab}_{\mathrm{Aut}\,V_\Lambda^+}(V_E^+)$. Let $g \in \mathrm{Stab}_{\mathrm{Aut}\,V_\Lambda^+}(V_E^+)$. Then $\xi(g)$ preserves E in Λ. Conversely, any element in Co_1 preserving E lifts to an element of $\mathrm{Stab}_{\mathrm{Aut}\,V_\Lambda^+}(V_E^+)$. Hence

$$1 \to \mathrm{Hom}(\Lambda,\mathbb{Z}_2) \to \mathrm{Stab}_{\mathrm{Aut}\,V_\Lambda^+}(V_E^+) \to \mathrm{Stab}_{Co_1}(E) \to 1$$

is exact. We note that $\mathrm{Stab}_{Co_1}(E)$ has the shape $2^{1+8}\cdot\Omega^+(8,2)$ and that its subgroup acting on E by ± 1 is the maximal normal 2-subgroup $O_2(\mathrm{Stab}_{Co_1}(E)) \cong 2^{1+8}$. Clearly, τ_e is a lift of the central element of $\mathrm{Stab}_{Co_1}(E)$ of order 2.

Let F be the subgroup of $\mathrm{Stab}_{\mathrm{Aut}\,V_\Lambda^+}(V_E^+)$ acting trivially on V_E^+. Then the following sequence is exact:

$$1 \to \{f_\alpha \in \mathrm{Hom}(\Lambda,\mathbb{Z}_2) \mid \langle \alpha, E\rangle \in 2\mathbb{Z}\} \to F \to O_2(\mathrm{Stab}_{Co_1}(E)) \to 1.$$

In particular, $F \cong 2^{16}.2^{1+8}$. Consider the orthogonal compliment B of E in Λ. Then F acts on V_B^+ and, the subgroup Q of F acting trivially on V_B^+ is described by the following exact sequence

$$1 \to \{f_\alpha \in \mathrm{Hom}(\Lambda,\mathbb{Z}_2) \mid \langle \alpha, B\oplus E\rangle \in 2\mathbb{Z}\} \to Q \to Z(\mathrm{Stab}_{Co_1}(E)) \to 1.$$

It is easy to see that $\{f_\alpha \in \mathrm{Hom}(\Lambda,\mathbb{Z}_2) \mid \langle \alpha, B\oplus E\rangle \in 2\mathbb{Z}\} = \{f_\alpha \mid \alpha \in E\} \cong 2^8$ and $Z(\mathrm{Stab}_{Co_1}(E)) \cong \mathbb{Z}_2$. Clearly, $\tau_e \in Q$ and $\tau_e \notin \mathrm{Hom}(\Lambda,\mathbb{Z}_2)$. Hence $Q \cong 2^9$. Note that $\mu^{-1}(Q) = \langle z, \tau_{\omega^+(\alpha)}, \tau_e \mid \alpha \in E\rangle \cong 2^{10}$. Since Q is normal in F, we obtain $F \cong 2^{9+16}$ and $\mu^{-1}(F) \cong 2^{10+16}$.

The group $\mathrm{Stab}_{\mathrm{Aut}\,V_\Lambda^+}(V_E^+)/F \cong 2^8.\Omega^+(8,2)$ acts faithfully on V_E^+, and we view this group as a subgroup of $\mathrm{Aut}\,(\hat{E})/\langle\theta\rangle \cong 2^8 : O^+(8,2)$. By Lemma 5.20, the stabilizer of τ_e in this group is isomorphic to $\mathrm{Aut}\,(E)'/\langle -1\rangle \cong \Omega^+(8,2)$. Thus we obtain the following exact sequence:

$$1 \to F \to \mathrm{Stab}_{\mathrm{Aut}\,V_\Lambda^+}(e) \to \mathrm{Aut}\,(E)'/\langle -1\rangle \to 1.$$

Hence $C_{\mathrm{Aut}\,V^\natural}(\tau_e, z) \cong \mu^{-1}(\mathrm{Stab}_{\mathrm{Aut}\,V_\Lambda^+}(e))$ has the structure $2^{10+16}.\Omega^+(8,2)$. Since z and τ_e are in the center, $C_{\mathrm{Aut}\,V^\natural}(\tau_e, z)$ has the shape $2^{2+8+16}.\Omega^+(8,2)$ □

Remark 5.22. On the setting of the theorem above, $C_{\mathrm{Aut}\,V^\natural}(\tau_e, z)$ acts on V_E^+ as $\Omega^+(8,2)$, which is the quotient of the commutator subgroup of the Weyl group of E_8 by its center. This result explains the $1A$ case of an observation by Glauberman and Norton, which describes some mysterious relations between the centralizer of z and some $2A$ elements commuting z in the Monster and the Weyl groups of certain sublattices of the root lattice of type E_8 (cf. Section 4 of [9]).

Remark 5.23. By [1, 13], an involution of the Monster is of type $2A$ if and only if it is a τ-involution. Hence if $e = \omega^{\pm}(\alpha)$, then z and τ_e generate a Klein's four group of type $(1A, 2A, 2A, 2B)$ in the Monster because $\langle \tau_e, z\rangle = \{1, \tau_{\omega^+(\alpha)}, \tau_{\omega^-(\alpha)}, z\}$ by Lemma 5.14. On the other hand, if $e = \varphi_x(e_E)$, then z and τ_e will generate a Klein's four group of type $(1A, 2A, 2B, 2B)$ because $\langle \tau_e, z\rangle = \{1, \tau_e, z\tau_e, z\}$ and $z\tau_e$ is not a τ-involution by Proposition 5.13.

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